

# Quantum Action Principle in Relativistic Mechanics

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A quantum version of the action principle is considered in the case of a free relativistic particle. The classical limit of the quantum action is obtained.

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## I. INTRODUCTION

In the works [1, 2] a new form of non-relativistic quantum mechanics in terms of a quantum action principle was proposed. The quantum action principle was formulated for a new object - a wave functional  $\Psi[x(t)]$  which, unlike a wave function  $\psi(x, t)$ , describes dynamics of a particle as a movement along a trajectory  $x(t)$ . The wave functional has the meaning of a probability density in the space of trajectories. It is this description of dynamics that is most appropriate for relativistic quantum mechanics. In the relativistic mechanics a trajectory of a particle must be replaced by an invariant geometrical object - a world line,  $x^\mu(\tau)$ , where  $\mu = 0, 1, 2, 3$  in the Minkowsky space. Here  $\tau$  is an arbitrary parameter along the world line. As a result, the wave functional  $\Psi[x^\mu(\tau)]$  becomes relativistic invariant preserving its probabilistic interpretation.

A special feature of relativistic mechanics is the presence of an additional symmetry - an independence of the action on the parametrization of a world line of a particle (see, for example, [3]). In ordinary quantum mechanics based on a wave function, it is necessary for its probabilistic interpretation to fix a time parameter by use of an additional gauge condition. In our approach the re-parametrization invariance must be unbroken. Gauge parameters which ensure invariance of the action must be added to a set of variational parameters of the quantum action principle. Therefore, the advantage of the new approach is the possibility of probabilistic interpretation of the quantum theory without a loss of its covariance. In the present paper the quantum action principle is considered in the case of a free relativistic particle. The classical limit of the quantum action is obtained.

## II. QUANTUM ACTION PRINCIPLE

We begin with the action of a particle in the geometrical form (the velocity of light is equal to 1):

$$I = -m \int ds, \quad (1)$$

where  $m$  is a mass of a particle, and

$$ds^2 = dx^2 \equiv dx^\mu dx_\mu \quad (2)$$

is the interval in the Minkowsky space. Here the Greece indices are lowered and risen by means of metric tensor with the signature  $(+, -, -, -)$ . Introducing an arbitrary parametrization of a world line,  $x^\mu = x^\mu(\tau)$ ,  $\tau \in [0, 1]$ , and defining a 4-vector of the canonical momentum,

$$p_\mu \equiv -m \frac{\dot{x}_\mu}{\sqrt{\dot{x}^2}}, \quad (3)$$

where the dot denotes the derivative with respect to the parameter  $\tau$ , one can write action (1) in the canonical form:

$$I = \int_0^1 \left( p_\mu \dot{x}^\mu - \chi H \right) d\tau. \quad (4)$$

Here  $\chi$  is a new variable which ensures the re-parametrization invariance of the action (4) and

$$H \equiv p^2 - m^2. \quad (5)$$

At this stage an invariant parameter along a world line can be introduced:

$$c(\tau) = \int_0^\tau \chi d\tau. \quad (6)$$

Then action (4) takes a form:

$$I = \int_0^C \left( p_\mu \dot{x}^\mu - H \right) dc, \quad (7)$$

where now the dot denotes the derivative with respect to the parameter  $c$ , and  $C \equiv c(1)$ .

Let us quantize action (7) following [1]. In the space of functionals  $\Psi[x^\mu(c)]$  we define a functional-differential operator:

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$$\hat{p}_\mu \equiv \frac{\tilde{\hbar}}{i} \frac{\delta}{\delta x^\mu(\tau)}, \quad (8)$$

where  $\tilde{\hbar}$  is a constant with the dimensionality  $Dj \cdot s^2$ . For an action operator  $\hat{I}$  which is obtained by substitution of (8) into (7), we consider the following secular equation:

$$\hat{I}\Psi \equiv \int_0^C \left[ \frac{\tilde{\hbar}}{i} x^\mu \frac{\delta \Psi}{\delta x^\mu} + \tilde{\hbar}^2 \frac{\delta^2 \Psi}{(\delta x^\mu)^2} + m^2 \Psi \right] dc = \lambda \Psi. \quad (9)$$

We formulate the quantum action principle as a condition of the extremum of this eigenvalue  $\lambda$  with respect to a set of parameters which will be defined below.

Let us introduce an exponential representation of the wave functional:

$$\Psi[x^\mu(c)] = \exp \left( \frac{\tilde{\hbar}}{i} \sigma[x^\mu(c)] + r[x^\mu(c)] \right), \quad (10)$$

where real functionals  $\sigma[x^\mu(c)]$  and  $r[x^\mu(c)]$  are supposed to be analytical. The latter means that they are representable by series in the degrees  $x^\mu(c)$ . With account of exponential representation (10), from Eq.(9) one obtains an eigenvalue,

$$\lambda = \int_0^C \left[ x^\mu \frac{\delta \sigma}{\delta x^\mu} - \left( \frac{\delta \sigma}{\delta x^\mu} \right)^2 + \tilde{\hbar}^2 \left( \left( \frac{\delta r}{\delta x^\mu} \right)^2 + \frac{\delta^2 r}{(\delta x^\mu)^2} \right) \right] dc + m^2 C, \quad (11)$$

and, in addition, a condition of its reality,

$$\int_0^C \left[ x^\mu \frac{\delta r}{\delta x^\mu} - 2 \frac{\delta \sigma}{\delta x^\mu} \frac{\delta r}{\delta x_\mu} - \frac{\delta^2 \sigma}{(\delta x^\mu)^2} \right] dc = 0. \quad (12)$$

Representation (11) is not final because eigenvalues must be independent on a world line  $x^\mu(c)$  except for boundary points  $b^\mu \equiv x^\mu(C)$  and  $a^\mu \equiv x^\mu(0)$  which are supposed to be fixed in the action principle. This demand imposes a set of differential equations on coefficients of series which are represented by the functionals  $\sigma[x^\mu(c)]$  and  $r[x^\mu(c)]$ . A solution of this set of equations depends on initial values of these coefficients at the moment  $c = 0$ . Therefore, we obtain an eigenvalue  $\lambda$  as a function of initial data. It is this function that must be stationary in the quantum action principle. The variable  $C$  also must be added to the set of variational parameters. In the next section, a quasi-classical approach for the quantum action principle will be considered, and the classical limit of the quantum action of a free relativistic particle will be obtained.

### III. CLASSICAL LIMIT OF QUANTUM ACTION

In the classical limit, the eigenvalue  $\lambda$  of the quantum action, Eq. (11), is completely defined by the functional  $\sigma[x^\mu(c)]$ . In the case of a free relativistic particle considered here, one can take into account only integral functionals in the following form:

$$\sigma[x^\mu(c)] = \int_0^C \left[ \sigma_{1\mu}(c) x^\mu(c) + \frac{1}{2} \sigma_2(c) (x)^2 + \dots \right] dc. \quad (13)$$

In the classical limit, one can consider functionals which are not higher than quadratic in  $x^\mu(c)$ . Substituting (13) into Eq.(11), after integration by parts one obtains the final form of the eigenvalue  $\lambda$ ,

$$\lambda = \left( \sigma_{1\mu} x^\mu + \frac{1}{2} \sigma_2 (x)^2 \right)_0^C - \int_0^C \sigma_2^2 dc + m^2 C, \quad (14)$$

and the condition of its independence on a world line  $x^\mu(c)$  in terms of two differential equations,

$$\dot{\sigma}_{1\mu} + 2\sigma_2 \sigma_{1\mu} = 0, \quad (15)$$

$$\dot{\sigma}_2 + 2\sigma_2^2 = 0. \quad (16)$$

A general solution of equations (15) and (16) is

$$\sigma_{1\mu}(c) = \frac{\sigma_{1\mu}^{(0)}}{1 + 2\sigma_2^{(0)}c}, \quad (17)$$

$$\sigma_2(c) = \frac{\sigma_2^{(0)}}{1 + 2\sigma_2^{(0)}c}, \quad (18)$$

where  $\sigma_{1\mu}^{(0)}$  and  $\sigma_2^{(0)}$  are initial values of the coefficients  $\sigma_{1\mu}$  and  $\sigma_2$ . Substitution of this solution into Eq.(14) gives the eigenvalue  $\lambda$  as a function of the initial data,  $\sigma_{1\mu}^{(0)}$ ,  $\sigma_2^{(0)}$ , and the invariant time parameter  $C$ :

$$\lambda = \sigma_{1\mu}^{(0)} \left( \frac{b^\mu}{1 + 2\sigma_2^{(0)}C} - a^\mu \right) + \frac{\sigma_2^{(0)}}{2} \left( \frac{(b^\mu)^2}{1 + 2\sigma_2^{(0)}C} - (a^\mu)^2 \right) - \frac{(\sigma_{1\mu}^{(0)})^2}{1 + 2\sigma_2^{(0)}C} + m^2 C. \quad (19)$$

It is this function that must be stationary with respect to the initial data,  $\sigma_{1\mu}^{(0)}$ ,  $\sigma_2^{(0)}$ , and the invariant time parameter  $C$  in the quantum action principle. The extremum condition with respect to the initial data gives

$$\sigma_{1\mu}^{(0)} = \frac{1}{2C} \left[ b_\mu - a_\mu \left( 1 + 2\sigma_2^{(0)} C \right) \right]. \quad (20)$$

Therefore, one of the initial data parameters,  $\sigma_2^{(0)}$  in this case, is not fixed in the classical limit of the quantum action principle, and the eigenvalue  $\lambda$  is degenerate. The extremum condition with respect to  $C$  gives

$$C = \pm \frac{1}{2m} \sqrt{(b-a)^2}. \quad (21)$$

Substituting (20), (21) into Eq.(19), one obtains the quantum action eigenvalue in the classical limit:

$$\lambda = \pm m \sqrt{(b-a)^2}. \quad (22)$$

This result coincides with classical action (1) calculated on the classical trajectory of a free particle. The wave functional corresponding to eigenvalue (22) has a phase which in the classical limit is proportional to:

$$\sigma[x^\mu] = \frac{1}{4} \int_0^Q (x^\mu(q) - \tilde{x})^2 dq, \quad (23)$$

where

$$\tilde{x}^\mu \equiv -\frac{b^\mu - e^Q a^\mu}{e^Q - 1}, \quad (24)$$

$$Q \equiv \ln \left( 1 + 2\sigma_2^{(0)} C \right). \quad (25)$$

#### IV. CONCLUSIONS

We conclude that in the classical limit the quantum action principle returns us to the original action of a relativistic particle calculated on a classical trajectory. Quantum corrections to this action will give us essential predictions of the new theory and define a new "Plank" constant  $\tilde{\hbar}$ . The parameter  $\sigma_1^{(0)}$ , which is indefinite in the classical limit, plays the role of a degree of excitation of a quantum particle.

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 [2] Natalya Gorobey, and Alexander Lukyanenko, arXiv: 0810.2255 (October 2008).

- [3] Michael B. Green, John H. Schwarz, and Edward Witten, *Superstring Theory* (Cambridge Univ. Press, N.Y., 1987).